Most of the formulas we have today are the result of mathematicians noticing patterns. The binomial theorem is one such pattern.

Expand $(a + b)^n$ for n = 0,1,2,3,4, &5

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^3 = a^3+3a^2b+3b^2a+b^3$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

$(a+b)^6 =$

powers a : 6 0

Powers b : 0 6

Exponents add : 6

1st 2 coefficients : 1, 6

last 2 " 6, 1

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & {}_0C_0 & & & \\
 & & {}_1C_0 & & {}_1C_1 & & \\
 & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & \\
{}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3
\end{array}
$$

Binomial Coefficient

The binomial coefficients that appear in the expansion of $(a+b)^n$ are the values of ${}_nC_r$ for r = 1, 2, 3, 4, ..., n.

A classical notation for ${}_nC_r$ is : $\begin{pmatrix} n \\ r \end{pmatrix}$ meaning "n choose r"

$${}_nC_r = \begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!} \qquad \begin{pmatrix} 15 \\ 11 \end{pmatrix}$$

${}_{15}C_{11}$

$= 1365$

Expand pascal's triangle to find the next row.

$(a+b)^0$ 1

$(a+b)^1$ 1 1

$(a+b)^2$ 1 2 1

$(a+b)^3$ 1 3 3 1

$(a+b)^4$ 1 4 6 4 1

$(a+b)^5$ 1 5 10 10 5 1

$(a+b)^6$ 1 6 15 20 15 6 1

$$1a^6b^0+6a^5b^1+15a^4b^2+20a^3b^3+15a^2b^4+6a^1b^5+1a^0b^6$$

Find the coefficient of x^{10} in the expansion of $(x + 2)^{15}$ $n=15$

$\binom{15}{10}x^{10}(2^5) = {}_{15}C_{10} \cdot 2^5 x^{10}$

$= \boxed{96,096}$

The Binomial theorem #67

For any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

where $\binom{n}{r} = {}_nC_r = \dfrac{n!}{r!(n-r)!}$

$$\left(x^2 + y\right)^3 =$$

$n=3$

$$\binom{3}{0}\left(x^2\right)^3 + \binom{3}{1}\left(x^2\right)^2(y)^1 + \binom{3}{2}\left(x^2\right)^1(y)^2 + \binom{3}{3}(y)^3$$

$$1x^6 + 3x^4y + 3x^2y^2 + 1y^3$$

$\left(3x - y^2\right)^5$ $\quad n=5$

$$\binom{5}{0}(3x)^5 + \binom{5}{1}(3x)^4(-y^2)^1 + \binom{5}{2}(3x)^3(-y^2)^2 + \binom{5}{3}(3x)^2(-y^2)^3 + \binom{5}{4}(3x)^1(-y^2)^4 + \binom{5}{5}(-y^2)^5$$

$$1\cdot 3^5x^5 - 5\cdot 3^4x^4y^2 + 10\cdot 3^3x^3y^4 - 10\cdot 3^2x^2y^6 + 5\cdot 3^1xy^8 - 1y^{10}$$

$$\boxed{243x^5 - 405x^4y^2 + 270x^3y^4 - 90x^2y^6 + 15xy^8 - y^{10}}$$

$$(-\#)^{\text{ODD}} = (-\#)$$

$$(-\#)^{\text{even}} = (+\#)$$