7.4 Partial Fractions

In 2.6 we learned how to break down a polynomial of higher degree using our factoring skills and synthetic division.

IE: Factor: $x^2+3x+2 = (x+1)(x+2)$

Breaking complicated expressions into less complicated expressions is a mathematicians most important tool. Today we are going to break down rational functions.

Write the sum as a single rational function

$$\frac{(x+3)}{(x+3)}\frac{3}{x-4}+\frac{2}{x+3}\frac{(x-4)}{(x-4)} = \frac{3(x+3)}{(x+3)(x-4)}+\frac{2(x-4)}{(x+3)(x-4)}$$

$$\frac{3x+9}{x^2-x-12}+\frac{2x-8}{x^2-x-12} = \frac{5x+1}{x^2-x-12}$$

Rational Functions

Unlike polynomials which break down into a **product** of factors, rational functions break down into a **sum** of fractions. This is called **Partial Fraction Decomposition**

$$\frac{3x-4}{x^2-2x} = \frac{2}{x} + \frac{1}{x-2}$$

Each fraction in the sum is called a partial fraction the entire sum is the partial fraction decomposition.

Find the partial fraction decomposition of:

$$\frac{5x-1}{x^2-2x-15} = \frac{A}{x+3} + \frac{B}{x-5}$$

$(x+3)(x-5)$ $\quad (x+3)(x-5)$ $\quad (x+3)(x-5)$ $\quad (x+3)(x-5)$

$5x-1 = A(x-5)+B(x+3)$

$5x-1 = Ax-5A+Bx+3B$

$5x-1 = Ax+Bx-5A+3B$

$5x-1 = (A+B)x+(-5A+3B)$

$A+B=5$

$-5A+3B=-1$

$$\begin{bmatrix}1 & 1\\ -5 & 3\end{bmatrix}\begin{bmatrix}A\\ B\end{bmatrix}=\begin{bmatrix}5\\ -1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\ -5 & 3\end{bmatrix}^{-1} = \frac{1}{3+5}\begin{bmatrix}3 & -1\\ 5 & 1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}3 & -1\\ 5 & 1\end{bmatrix} = \begin{bmatrix}3/8 & -1/8\\ 5/8 & 1/8\end{bmatrix}\begin{bmatrix}5\\ -1\end{bmatrix} = \begin{bmatrix}\frac{15}{8}+\frac{1}{8}\\ \frac{25}{8}-\frac{1}{8}\end{bmatrix} = \begin{bmatrix}2\\ 3\end{bmatrix}$$

$$\boxed{\frac{2}{x+3}+\frac{3}{x-5}}$$

Find the partial fraction decomposition of:

$$\frac{3(x-1)}{x^2+5x+4} = \frac{A}{x+4} + \frac{B}{x+1}$$

$(x+4)(x+1)$

$3(x-1) = A(x+1) + B(x+4)$

$3x-3 = Ax + A + Bx + 4B$

$3x-3 = x(A+B) + (A+4B)$

$A+B=3$

$A+4B=-3$

$$\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^{-1} = \frac{1}{4-1}\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 4+1 \\ -1-1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$\frac{5}{x+4} + \frac{-2}{x+1}$$

Find the partial fraction decomposition of:

$$\frac{7}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$7 = A(x-3) + B(x-2)$

$7 = Ax - 3A + Bx - 2B$

$7 = x(A+B) + (-3A-2B)$

$A+B=0$

$-3A-2B=7$

$$\begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}^{-1} = \frac{1}{-2+3}\begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \end{bmatrix}$$

$$\boxed{\frac{-7}{x-2} + \frac{7}{x-3}}$$